his *Anleitung zur Algebra.*[1] During the eighteenth century the Naperian definition, based on the two progressions, continued to be the prevailing definition. As late as 1808, C. F. Kaussler expressed decided preference for it, the new definition offering to beginners "gaps and obscurities."[2]

We have seen that Euler extended the exponential concept even to the use of imaginary exponents. We have seen that the expression $\sqrt{-1}^{\sqrt{-1}}$ is associated with his name; he recognized that it was infinitely many-valued. When the exponent is real and fractional, the multiple values of $a^{m/n}$ were recognized by Euler and others in research articles. As a rule, this many-valuedness was not discussed at all in text-books of the eighteenth century, such as the algebras of Saunderson, Blassière and Lacroix, or the mathematical texts of La Caille, J. F. Häseler, Abbé Sauri, Karsten, Reyneau, Bézout, Gherli, Rivard and Bertrand, or the mathematical dictionaries of E. Stone and Ch. Wolf.

The relation $a = e^{\log a}$ was easily deduced from the new definition of logarithms and was used by some writers of this century, such as G. Fontana in 1782.

[Further instalments will follow, carrying this history through the nineteenth century. EDITORS.]

SOME INVERSE PROBLEMS IN THE CALCULUS OF VARIATIONS.

By E. J. MILES, Yale University.

In the so-called inverse problem[1] of the calculus of variations a doubly infinite family of curves

$$y = f(x, \alpha, \beta) \tag{1}$$

is given and it is required to find a function $F(x, y, y')$ such that the given system of curves forms the system of extremals for the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx. \tag{2}$$

Darboux[2] has shown that the problem always has an infinite number of solutions which can be reduced to quadratures. For if in the Euler differential equation

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0$$

the expression $G(x, y, y')$, where $y'' = G(x, y, y')$ is the differential equation

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[1] L. Euler, *Anleitung zur Algebra*, St. Petersburg, 1770, 1. Theil, Cap. 21.

[2] C. F. Kaussler, *Lehre von den Logarithmen*, Tübingen, 1808, preface. Quoted by Tropfke Gesch. d. Elem.-Math., 2. Band, Leipzig, 1903, p. 142.

[1] For a treatment of this problem see for instance Bolza: *Vorlesungen über Variationsrechnung*, § 6c.

[2] Darboux: *Theorie des Surfaces*, vol. III, Nos. 604, 605.

from which the curves (1) are obtained, is substituted for y'' then the resulting equation

$$F_y - F_{y'x} - y'F_{y'y} - G(x, y, y')F_{y'y'} = 0 \tag{3}$$

is satisfied identically in x, y and y'. Differentiating (3) as to y' and placing $M = F_{y'y'}$ a linear partial differential equation of first order is obtained, viz:

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + G\frac{\partial M}{\partial y'} + G_{y'}M = 0. \tag{4}$$

From this M may be obtained and then F by two quadratures.

Few examples illustrating this theory have been given. Hamel[1] has studied the case when the given curves are straight lines and Stromquist[2] the case when they are circles with center on the x-axis. The present note is intended to add further illustrative examples by considering first the case when the given curves are catenaries, secondly the case when the given curves are parabolas whose axes coincide with the x-axis, and finally the case when they are the ellipses or hyperbolas $x^2/\alpha + y^2/\beta = 1$.

§ 1. When the Given Curves are Catenaries.

It will be assumed that the catenaries have the x-axis for directrix. They therefore have the equation

$$y = \alpha \cosh \frac{x - \beta}{\alpha}.$$

The differential equation[3] from which this equation is obtained is

$$y'' = \frac{1 + y'^2}{y}.$$

Substituting this value of y'' the Euler equation then gives the following identity in x, y and y'

$$F_y - F_{y'x} - y'F_{y'y} - \frac{1 + y'^2}{y}F_{y'y'} = 0. \tag{5}$$

Hence M in this case must be obtained from the partial differential equation

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{1 + y'^2}{y}\frac{\partial M}{\partial y'} + \frac{2y'}{y}M = 0. \tag{6}$$

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[1] Hamel: *Über die Geometrien in denen die Geraden die Kurzesten sind;* dissertation, Göttingen, 1901; *Mathematische Annalen*, vol. 57 (1903), pp. 231–264.

[2] Stromquist: "*On geometries in which circles are the shortest lines,*" *Transactions of the American Mathematical Society*, vol. 7 (1906), pp. 175–183.

[3] This is readily verified when it is remembered that the projection of the ordinate of any point of the catenary on the normal to the catenary at this point is constant. Stated analytically this says that $y/\sqrt{1 + y'^2} = \alpha$, whence the above result.

To integrate this place $M = e^z$ and equation (6) becomes

$$\frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + \frac{1+y'^2}{y} \frac{\partial z}{\partial y'} + \frac{2y'}{y} = 0. \tag{7}$$

Then if Φ_1 denotes an arbitrary function of the arguments indicated, the most general solution of equation (7) is found to be

$$\Phi_1 \left\{ \frac{y}{\sqrt{1+y'^2}},\quad x - \frac{y}{\sqrt{1+y'^2}} \log\left(\sqrt{1+y'^2} \pm y'\right),\quad z + \log\left(1+y'^2\right) \right\} = 0$$

or

$$z + \log(1+y'^2) = \Phi_2 \left\{ \frac{y}{\sqrt{1+y'^2}},\quad x - \frac{y}{\sqrt{1+y'^2}} \log\left(\sqrt{1+y'^2} \pm y'\right) \right\}.$$

Hence M assumes the form

$$M = \frac{1}{1+y'^2}\Phi \left\{ \frac{y}{\sqrt{1+y'^2}},\quad x - \frac{y}{\sqrt{1+y'^2}} \log\left(\sqrt{1+y'^2} \pm y'\right) \right\},$$

where again the function Φ is an arbitrary function of its arguments.

From this it follows that a first integration gives

$$\begin{aligned}\frac{\partial F}{\partial y'} &= \int \frac{1}{1+y'^2}\Phi \left\{ \frac{y}{\sqrt{1+y'^2}},\quad x - \frac{y}{\sqrt{1+y'^2}} \log\left(\sqrt{1+y'^2} \pm y'\right) \right\} dy' + \lambda(x, y) \\ &= \int_0^{y'} \frac{1}{1+t^2}\Phi \left\{ \frac{y}{\sqrt{1+t^2}},\quad x - \frac{y}{\sqrt{1+t^2}} \log\left(\sqrt{1+t^2} \pm t\right) \right\} dt + \lambda(x, y).\end{aligned}$$

Finally on integrating a second time F is found to be of the form

$$\begin{aligned}F = \int_0^{y'} (y'-t) \frac{1}{1+t^2}\Phi \left\{ \frac{y}{\sqrt{1+t^2}},\quad x - \frac{y}{\sqrt{1+t^2}} \log\left(\sqrt{1+t^2} \pm t\right) \right\} dt \\ + \lambda(x, y)y' + \mu(x, y),\end{aligned} \tag{8}$$

in which λ and μ are arbitrary functions of x and y.

Now every F which satisfies equation (5) necessarily satisfies equation (6) but not conversely. It is therefore necessary to find what conditions the functions λ and μ must satisfy in order that the function F given by (8) may satisfy the equation (5). When this value of F is substituted in equation (5) it must result in a function independent of y'. Consequently it may be evaluated by giving y' any definite value, say $y' = 0$. This gives us the relation

$$-\frac{1}{y}\Phi(y, x) + \mu_y - \lambda_x = 0. \tag{9}$$

If then $\bar{\lambda}$ and $\bar{\mu}$ are particular values of λ and μ which satisfy condition (9), the most general values of λ and μ are

$$\lambda = \bar{\lambda} + \frac{\partial V}{\partial y}, \quad \mu = \bar{\mu} + \frac{\partial V}{\partial x},$$

where V is an arbitrary function of x and y. If one of the functions $\bar{\lambda}$ and $\bar{\mu}$, say $\bar{\mu}$, is chosen as zero, equation (9) gives

$$\frac{\partial \bar{\lambda}}{\partial x} = -\frac{1}{y}\Phi(y, x),$$

and therefore

$$\bar{\lambda} = -\frac{1}{y}\int_{x_0}^{x_1} \Phi(y, x)dx.$$

Hence the function F has as its final form the following:

$$F = \int_0^{y'} (y' - t)\frac{1}{1+t^2}\Phi\left\{\frac{y}{\sqrt{1+t^2}},\quad x - \frac{y}{\sqrt{1+t^2}}\log\left(\sqrt{1+t^2} \pm t\right)\right\}dt$$
$$- \frac{y'}{y}\int_{x_0}^{x}\Phi(y, x)dx + y'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

This is the most general form which it can assume and have as extremals the given set of catenaries.

Suppose now that a further restriction is made, namely, that the extremals are to be perpendicular to their transversals. In order to assertain what the restriction is in this case it is only necessary to recall the general condition of orthogonality of extremals and transversals, viz:

$$F(x, y, y') = \psi(x, y)\sqrt{1 + y'^2}.$$

In case of the present problem this leads to the condition

$$\Phi\left\{\frac{y}{\sqrt{1+y'^2}},\quad x - \frac{y}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right)\right\} = \frac{\psi(x, y)}{(1 + y'^2)^{\frac{1}{2}}}.$$

But in order that there may be some relation between the expression $\psi(x, y)/\sqrt{1 + y'^2}$ and the arguments of Φ it is necessary that their Jacobian vanish identically in x, y and y', i. e.,

$$\begin{vmatrix} \dfrac{\psi_x}{\sqrt{1+y'^2}} & \dfrac{\psi_y}{\sqrt{1+y'^2}} & -\dfrac{y'\psi}{(\sqrt{1+y'^2})^3} \\ 0 & \dfrac{1}{\sqrt{1+y'^2}} & -\dfrac{yy'}{(\sqrt{1+y'^2})^3} \\ 1 & -\dfrac{1}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right) & \dfrac{yy'}{(\sqrt{1+y'^2})^3}\log\left(\sqrt{1+y'^2} \pm y'\right) \mp \dfrac{y}{1+y'^2} \end{vmatrix} \equiv 0.$$

Expanding this determinant and grouping like powers of y' it is found that the following equation results:

$$\pm y\psi_x + y'[y\psi_y - \psi] \equiv 0.$$

Hence

$$y\psi_x = 0, \qquad y\psi_y - \psi = 0$$

and therefore

$$\psi = ky.$$

Hence if the extremals are to be catenaries which are orthogonal to their transversals the function Φ must assume the form

$$\frac{ky}{\sqrt{1 + y'^2}}.$$

§ 2. The Case of Parabolas.[1]

Let the equation of the parabolas be taken in the form

$$y^2 = 2\alpha x + \beta.$$

Then

$$y'' = -\frac{y'^2}{y}$$

and equation (3) becomes in this case

$$F_y - F_{y'x} - y'F_{y'y} + \frac{y'^2}{y} F_{y'y'} = 0. \tag{10}$$

Hence M is determined by the following equation:

$$\frac{\partial M}{\partial x} + y' \frac{\partial M}{\partial y} - \frac{y'^2}{y} \frac{\partial M}{\partial y'} - \frac{2y'}{y} M = 0. \tag{11}$$

Integrating by the method of the preceding section it is easily verified that the most general solution of equation (11) is

$$M = y^2\Phi(yy', \; y^2 - 2xyy'),$$

where again Φ is an arbitrary function of its arguments.

Hence

$$\frac{\partial F}{\partial y'} = \int_0^{y'} y^2\Phi(yt, \; y^2 - 2xyt)dt + \lambda(x, y)$$

and therefore

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[1] Only points in the upper half plane are considered, for the tangent at the vertex of the parabola $y^2 = 2\alpha x + \beta$ is infinite and this case cannot be treated by the usual methods of the calculus of variations when the curve is taken in the form $y = y(x)$. If then $x_0 \neq x_1$, and $y_0 \neq y_1$, there passes one and but one parabola of the above type through any two points in the upper half plane.

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt, y^2 - 2xyt)dt + y'\lambda(x, y) + \mu(x, y),$$

where λ and μ are functions of x and y.

When this value of F is substituted in equation (10) giving an equation independent of y' it is found upon evaluating by assigning the value zero to y' that λ and μ must satisfy the relation

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\mu}{\partial y}.$$

Hence the most general expressions for λ and μ are

$$\lambda = \frac{\partial V}{\partial y}, \quad \mu = \frac{\partial V}{\partial x},$$

V being an arbitrary function.

The most general solution of the problem is therefore

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt', y^2 - 2xyt)dt + y'\frac{\partial V(x, y)}{\partial y} + \frac{\partial V(x, y)}{\partial x}.$$

§ 3. When the Given Curves are Ellipses or Hyperbolas.

In this case the equations of the curves will be taken in the form[1]

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1.$$

The differential equation belonging to these curves is found to be

$$y'' = \frac{yy' - xy'^2}{xy}.$$

Hence the partial differential equation from which M is obtained is

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{yy' - xy'^2}{xy}\frac{\partial M}{\partial y'} + \frac{y - 2xy'}{xy}M = 0. \tag{12}$$

By the process previously described the following expression is found for M

$$M = \frac{y}{x(y - xy')}\Phi\left(\frac{-y'}{x(y - xy')}, \frac{1}{y(y - xy')}\right).$$

[1] Here again only points in the upper half plane are considered. Then if the point (x_1, y_1) is not on either of the lines

$$x + \frac{x_0}{y_0}y = 0, \quad x - \frac{x_0}{y_0}y = 0,$$

it follows that there is a unique determination of the constants $1/\alpha$ and $1/\beta$. Hence, barring these exceptional cases, it follows that through any two points of the upper half plane there passes one ellipse or one hyperbola defined by the equation $x^2/\alpha + y^2/\beta = 1$.

Hence by quadrature

$$F = \int_0^{y'} (y' - t)\frac{y}{x(y - xt)}\Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right)dt + y'\lambda(x, y) + \mu(x, y).$$

It is easily proved that the functions λ and μ must again satisfy the condition

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\mu}{\partial y}.$$

Therefore the most general solution for F is seen to be

$$F(x, y, y') = \int_0^{y'} (y' - t)\frac{y}{x(y - xt)}\Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right)dt + y'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

Furthermore it can be proved that in neither of the last two cases can the function Φ be so chosen that the transversals are perpendicular to the extremals.

THE PROBABILITY INTEGRAL DEDUCED BY MEANS OF DEVELOPMENTS IN FINITE FORM.

By EDWARD L. DODD, University of Texas.

In modern mathematics, expansions in *finite* form are constantly coming into greater favor in preference to infinite series from which the terms after the first two or three have been dropped.

The object of this paper is to express in modern form the deduction of the probability integral as given by J. Bertrand,[1] whose method follows Laplace's,[2]—though his integral is not quite the same.

Let p be the probability that an event will happen in a single trial, where $0 < p < 1$; and let $q = 1 - p$. Then the probability, P_r, that the event will happen exactly r times on s trials is

$$P_r = \frac{s!}{r!\,(s - r)!}p^r q^{s-r}. \tag{1}$$

Here r is a positive integer not greater than s, or r is zero, accepting unity as the value of $0!$

Now let

$$l = sp - r. \tag{2}$$

Then $r = sp - l$, and

$$P_r = \frac{s!}{(sp - l)!(sq + l)!}p^{sp-l}q^{sq+l}. \tag{3}$$

[1] *Calcul des Probabilités* (1889), p. 76.

[2] *Œuvres Complètes*, VII, Theorie Analytique des Probabilités, p. 284.